PLANAR PSEUDO-TRIANGULATIONS, SPHERICAL PSEUDO-TILINGS AND HYPERBOLIC VIRTUAL POLYTOPES

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ABSTRACT. We wish to draw attention to an interesting and promising interaction of two theories.

On the one hand, it is the theory of **pseudo-triangulations** which was useful for implicit solution of the carpenter's rule problem and proved later to give a nice tool for graph embeddings.

On the other hand, it is the theory of **hyperbolic virtual polytopes** which arose from an old uniqueness conjecture for convex bodies (A. D. Alexandrov's problem): suppose that a constant C separates (non-strictly) everywhere the principal curvature radii of a smooth 3-dimensional convex body K. Then K is necessarily a ball of radius C.

The two key ideas are:
- Passing from planar pseudo-triangulations to spherical pseudo-tilings, we avoid non-poited vertices. Instead, we use pseudo-di-gons. A theorem on spherically embedded Laman-plus-one graphs is announced.
- The difficult problem of hyperbolic polytopes constructing can be reduced to finding spherically embedded graphs.

0. INTRODUCTION

Pseudo-triangulations are opposite to the traditional planar graph embeddings - they are as non-convex as possible.

As a parallel phenomenon, hyperbolic virtual polytopes are opposite to convex polytopes: convexity is replaced by saddle property.

The two theories have a nice interaction, which is demonstrated in the paper.

Even at first glance one can see that pseudo-triangulations look very much like the fans of hyperbolic virtual polytopes. Indeed, in both cases we have a pointed tiling. But whereas pseudo-triangulations are planar drawings, the fans of hyperbolic virtual polytopes are spherical ones.

However, the relationship is much deeper. As is shown in the paper, some items are absolutely the same, some are easily adjustable, but some are different.

In the first two sections, we sketch very briefly the two theories, omitting all details and applications and referring the reader to detailed papers from the list of references.

In the third section, we bring the two theories together.

We show that for spherical embeddings, the usage of pseudo di-gons allows pointed embeddings not only for Laman graphs, but also of graphs with a greater number of edges. Even some Laman-plus-k graphs (for any natural number k) admit a pointed embedding (see Example 3.7 for $k = 5$). In particular, we announce the following theorem on Laman-plus-one graph embedding. It is parallel to the results of [7],[25] (quotated also in section 1).

Theorem on spherical embedding of Laman-plus-one graphs. (Theorem 3.3)

- Each Laman-plus-one graph admits a straightened embedding (all the edges are geodesic segments) in the 2-dimensional sphere S^2 such that
  (1) it generates a pointed pseudo-tiling of the sphere.
  (2) The tiles are either pseudo-triangles or pseudo-di-gons.
  (3) The number of pseudo-di-gons equals 4.
  (4) The embedding can be constructed inductively, via geometric Henneberg constructions starting from the fan of a hyperbolic tetrahedron (i.e. from the pointer embedding of K_4, see Fig. 1)
- Any spherical embedding of a Laman-plus-one graph as a nice pseudo-tiling has exactly 4 pseudo di-gons.
- If a graph admits a straightened embedding possessing the above properties 1-3, then it is a Laman-plus-one graph.   □

Some very natural questions on spherical pointed tilings are formulated.

A simple example (Example 3.9) demonstrates how the methods of pseudo-triangulations work for the sake of the hyperbolic polytopes, and visa versa.

At the end of Section 4, we arrange the parallel terms from both the theories in a kind of a dictionary. The correspondence of the objects is not always straightforward, but we find better to skip additional technicalities in order to stress similarity of the ideas.

Acknowledgments

1. PSEUDO-TRIANGULATIONS

Topological preliminaries.

In the paper, we consider only planar graphs.

Let Γ be a graph with v vertices and e edges.

Γ is a *Laman graph* if $e = 2v - 3$ and every subset of k vertices spans at most $2k - 3$ edges.

Laman graphs are interesting because they are *minimally rigid graphs* - their generic embeddings (as bar-and-joint frameworks) are rigid.

Γ is *Laman-plus-one graph* (respectively, *Laman-plus-k*) if there is an edge (respectively, k edges) such that after its removing the graph becomes a Laman graph.

Minimal Laman-plus-one graphs are called *rigidity circuits*. A generic embedding of a rigidity circuit possesses a unique (up to a constant) non-vanishing self-stress. Therefore, the embedded graph has a 3D lift (i.e. it can be represented as a projection of a spatial polytope). Besides, such a graph has a unique (up to a homothety) *Maxwell reciprocal* (a 3-dimensional dual to the graph polytope constructed by the stress).

Laman and Laman-plus-one graphs admit an inductive construction starting with elementary graphs. At each step, a new vertex is added via one of the following two *Henneberg constructions*.

- Henneberg 1 construction: add a new vertex, connecting it via two new edges to two old vertices.
- Henneberg 2 construction: add a new vertex inside an old edge (and thus split the edge into two new ones), connecting the new vertex by another new edge to a third vertex.

Lemma 1.1. *(see [7], [28])*

(1) *A graph is Laman if and only if it admits an inductive construction starting with a graph with two vertices and one edge.*

(2) *A graph is Laman-plus-one if and only if it admits an inductive construction starting with K_4 (the complete graph with 4 vertices).*

   In both cases, each step is a Henneberg construction. □

Geometric realizations.

Consider a planar embedding of a graph Γ. We say that its vertex is *pointed* if one of the adjacent angles is greater than π.

An embedding is *pointed* if all its vertices are pointed.

A *pseudo-triangle* is a simple polygon (a non-crossing planar broken line) which has exactly 3 convex vertices.

A *pseudo-triangulation* is a tiling of a convex polygon in the plane such that each tile is a pseudo-triangle.

A pseudo-triangulation is *pointed* if all its vertices are pointed.

A pseudo-triangulation is *pointed-plus-one* if all its vertices, except for exactly one vertex, are pointed.

Theorem 1.2 (7)**.** *(On planar embedding of Laman and Laman-plus-one graphs)*

- *A graph Γ is a planar Laman graph if and only if it can be embedded in the plane as a pointed pseudo-triangulation.*
- *A graph Γ is a planar Laman-plus-one graph if and only if it can be embedded in the plane as a pointed-plus-one pseudo-triangulation.*
   In both cases the embeddings can be constructed inductively such that
- *Construction starts by an embedding of one-edge-graph (for Laman graphs) or by an embedding of K_4 (for Laman-plus-one graphs)*

- *On each step, we get a pointed pseudo-triangulation (respectively, a pointed-plus-one pseudo-triangulation)*
- *Each step is a geometric realization of a Henneberg construction. The construction is local: it does not change the positions of old vertices.* □

2. Hyperbolic virtual polytopes

Roughly speaking, virtual polytopes are geometric realizations of Minkowski difference of convex polytopes.

They were introduced originally by A. Pukhlikov and A. Khovanskij in [10], appeared also in a different disguise in the polytope algebra of P. McMullen [14].

Hyperbolic virtual polytopes [18-20] are virtual polytopes with special saddle properties.

In the section, we try to give a shortcut to necessary notions.

Convex polytopes in $\mathbb{R}^3$ form a semigroup $\mathcal{P}$ with respect to the Minkowski addition $\otimes$.

The semigroup $\mathcal{P}$ is isomorphic to the semigroup of continuous convex piecewise linear (with respect to a *fan*) functions defined on $\mathbb{R}^3$.

The isomorphism maps a convex polytope to its support function.

(A necessary reminding: the support function of a polytope is piecewise linear with respect to some conical tiling of the space. To visualize the tiling, we intersect it with a unite sphere centered at O and get a spherical *fan* of the polytope. It is a spherical tilings, all tiles are convex. In some sense, a polytope can be considered as the Maxwell's reciprocal of its fan.)

Passing to the Grothendieck group $\mathcal{P}^*$ (it is the group of formal differences of convex polytopes) which is called the *group of virtual polytopes*, only the convexity property disappears. Thus we get a group isomorphism

virtual polytope $\longleftrightarrow$ continuous piecewise linear (with respect to a fan) function defined on $\mathbb{R}^3$.

The skeleton of the fan has a self-stress. A virtual polytope (it can be considered as the Maxwell's reciprocal of its fan) can be represented geometrically as a polytopal function [14] or as a closed polytopal surfaces [20].

We don't mind (and can not avoid) self-crossing 3D reciprocals. This is because hyperbolic polytopes (considered as spatial piecewise linear surfaces) are necessarily self-crossing (except for degenerated cases as hyperblic tetrahedron), see Example 2.3.

Given a virtual polytopes, the tiles of its fan can be non-convex.

Recall that the support function of a convex polytope is convex, i.e. its graph is a convex surface (it is reasonable to consider either the spherical graph or the collection of affine graphs [18],[19]).

Among virtual polytopes we point out the class of hyperbolic virtual polytopes.

Definition 2.1. A virtual polytope is hyperbolic if the graph of its support function is a saddle surface.

This definition arose quite natural from the following conjecture.

Given a smooth 3-dimensional convex body K and a constant C such that $R_1 \leq C \leq R_2$ holds at each point of ∂K (R_1 and R_2 stand for the principal curvature radii of K), the body K is necessarily a ball of radius C.

The conjecture proved to be wrong (see [12], [18]), and here is a way of constructing counterexamples (which are unexpectedly diverse).

- Construct a hyperbolic polytope (this is the most difficult step, for hyperbolic polytopes are very rare phenomena among virtual polytopes)
- Smoothen its support function h (preserving saddle property)
- Add to h the support function of a ball (which is sufficiently large to make the sum convex). The result is the support function of a counterexample to the conjecture.

In the framework of the theory of hyperbolic polytopes, pointed spherical tilings appear due to the following simple observation.

Lemma 2.2. *(see [20])*

- *The fan of a virtual polytope K is a pointed tiling $\Rightarrow K$ is hyperbolic.*
- *If K is simplicial, then*
  the fan of K is a pointed tiling $\Leftrightarrow K$ is hyperbolic. $\qquad\square$

Example 2.3. *Figure 1 presents the fan of the hyperbolic tetrahedron. It is the simplest hyperbolic polytope.*

The hyperbolic tetrahedron is useless for the above conjecture (for this polytope, the smoothing technique of [18] does not work), but it proved to be the starting point for Laman-plus-one graphs embeddings (Theorem 3.3).
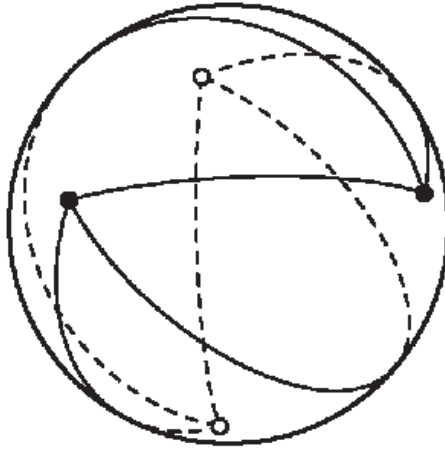
FIGURE 1.

Example 2.4. *Figure 2 depicts a hyperbolic polytope (viewed as a complicated self-intersecting 3D surface) and its fan.*

The hyperbolic polytope has 8 horns - the non-saddle vertices. By duality, they correspond to 8 pseudo di-gons (see the definition in Section 3).

The di-gons are marked grey. Note that only half of each of di-gons is visible.

*For each hyperbolic polytope, **horns are dual to pseudo-di-gons**. For a simplicial hyperbolic polytope, duality maps bijectively horns of the polytope to the pseudo-di-gons of its fan [20].*
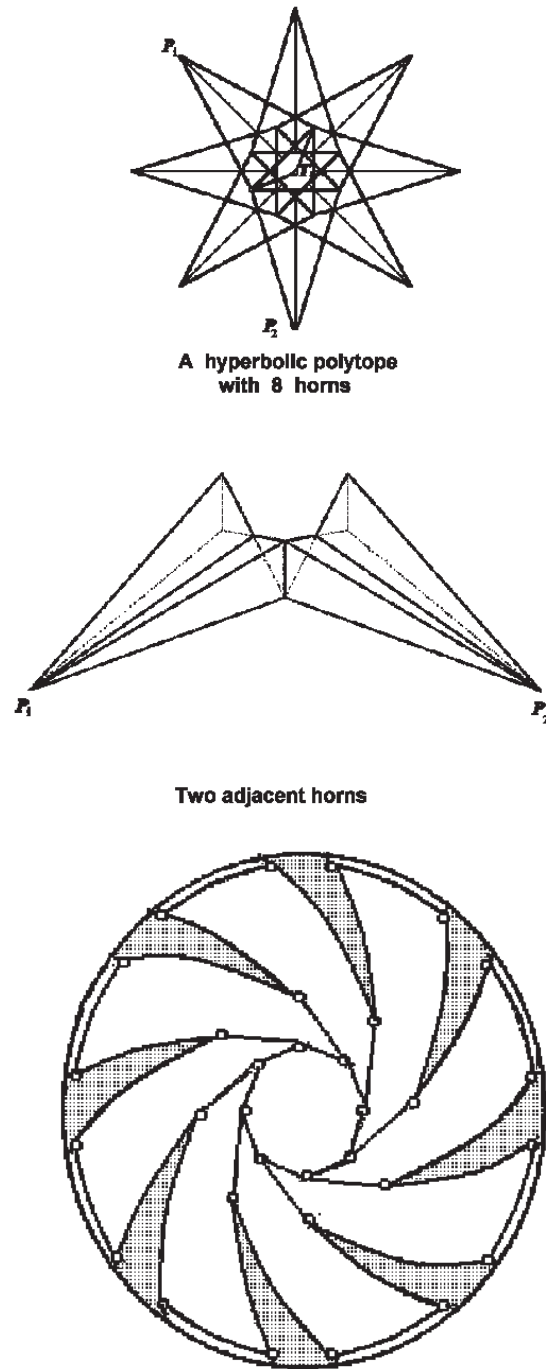
A hyperbolic polytope
with 8 horns

Two adjacent horns

FIGURE 2.

It makes sense to color the edges of the fan of a hyperbolic polytope K red and blue. The support function of K is concave up along the red edges and concave down on the blue ones.

The theory of hyperbolic polytopes has another nice applications. Here we list some problems from classical geometry which have been solved using this theory.

(1) A refinement of A.D. Alexadrov's unieqness theorem for 3D-polytopes with non-insertable pairs of parallel faces [19].
(2) Extrinsic geometry of saddle surfaces with injective Gaussian mapping [18-21].
(3) Isotopy problem for saddle surfaces [21].

3. Putting the pieces together

¿From now on, we consider graph embeddings in the sphere S^2.

The first key idea is: for the sake of hyperbolicity, we avoid non-poited vertices. Instead, we use pseudo-di-gons.

A *pseudo di-gon* is a spherical polygon (a non-crossing broken line embedded in the sphere with a fixed interior domain; all segments are geodesic segments) which has exactly two convex vertices.

Definition 3.1. A ***nice pseudo-tiling*** is a spherical tiling which is
- pointed
- each tile is either a (spherical) pseudo-triangle or a pseudo-di-gon.

The di-gons of a nice pseudo-tiling are of a particular interest from the viewpoint of both theoties. Firstly, they correspond by duality to the horns of hyperbolic polytopes. Secondly, their number determins the Laman-type counts.

Proposition 3.2. *The number of di-gons can range from 0 to infinity.*

Proof. To construct an embedded graph with $k > 3$ di-gons, take the fan of a hyperbolic polytope with k horns (see [20]) and find its nice pseudo subtiling. For $k < 4$, it is easy. $\square$

Theorem 3.3.   • *Each Laman-plus-one graph admits a straightened embedding (all the edges are geodesic segments) in the 2-dimensional sphere S^2 such that*
  (1) *it generates a nice pseudo-tiling of the sphere;*
  (2) *there are exactly 4 pseudo di-gons;*
  (3) *the embedding can be constructed inductively, via geometric Henneberg constructions starting from the fan of the hyperbolic tetrahedron (i.e. from a pointer embedding of K_4, see Fig. 1).*
- *Any embedding of a Laman-plus-one graph as a nice pseudo-tiling has exactly 4 pseudo di-gons.*
- *If a graph admits a straightened embedding possessing the above properties 1-2, then it is a Laman-plus-one graph.*

Proof (a sketch).

1. The proof repeats that of Theorem 3.2, [7].

Two items are essential.

On one hand, dislike [7], we do not care about geometric realization of a combinatorial pseudo-triangulation (we do not prescribe what angles should be greater than π).

On the other hand, the geometric shape of spherical pseudo-triangles and pseudo-di-gons can be bad and can cause obstacles for applying Henneberg constructions.

This motivates the following definition.

A pseudo-triangle (or a pseudo di-gon) is called *H-good* if it admits geometricaly any Henneberg construction.

The H-goodnes can be expressed in terms of *feasible regions* (see [7]) of the tile.

All planar pseudo-triangles (and therefore, all their spherical images) are good (as is proven in [7]).

When applying a geometrical Henneberg construction, it is always possible to preserve H-goodnes of all the tiles.

Thus we get an algorithm of the desired embedding (which is nearly the same as in [7]):

- By Lemma 1.1, we have an inductive topological Henneberg construction of the graph starting from K_4.
- Take the pointed embedding of K_4 (Fig. 1). It is H-nice.
- Realize geomerticaly step by step the Henneberg constructions, preserving H-goodness. Note that the number of pseudo di-gons does not change.

2. The proof repeats literally the corners counts from [7] and recalls very much color changes counts for hyperbolic fans [20].

Denote by v the number of vertices, by c the number of *corners* (i.e. the number of angles which are greater than π), by e the number of edges, by f_3 the number of pseudo-triangles, and by f_2 the number of pseudo di-gons.

We have $v - e + f_2 + f_3 = 2$ (Euler formula),

$e = 2v - 2$ (Laman-plus-one count),

$c = 2f_2 + 3f_3$ (obvious),

and $c = 2e - v$ (corners count), which easily complete the proof.

3. Assume that an embedding of a graph Γ generates a nice pseudotiling with 4 pseudo-di-gons.

The above counts imply that $e = 2v - 2$.

Fix k vertices and denote by Γ' the spanned subgraph. It generates a pointed spherical tiling. Then the number of di-gons is not greater than 4 because no di-gon admits a pointed subtiling into a collection of pseudo-triangles. Similar counts complete the proof. $\qquad\square$

Question 3.4. *What part of combinatorics of a Laman-plus-one graph embedding can be prescribed (as is done in [7])?*

Question 3.5. *Is there any canonical Laman graphs embedding in the sphere?*

(Note that we have already at least two different pointed spherical embeddings for a Laman graph: the first one comes from its planar pointed embedding raised to the sphere; to get the second one, just add an edge, embed the result according to Theorem 3.3, and then erase the edge.)

Question 3.6. *There exist nice pseudo-tilings with no di-gons (all tiles are pseudo-triangles). What is a characterization of the set of planar graphs admitting such an embedding?*

Example 3.7. *The tiling from Example 2.5 obviously admits a subtiling which is a nice pseudo-tiling. It gives a pointed embedding of a graph with $e = 2v + 2$.*

Thus manipulations with the number of di-gons enable us to embed graphs with many edges.

Question 3.8. *Is each nice pseudotiling such that the number of pseudo-di-gons equals $3 + k$ generated by a Laman-plus-k graph?*

Another key idea: the difficult problem of constructing of hyperbolic polytopes (3D objects) can sometimes be reduced to constructing embedded graphs (2D objects).

The following simple example demonstrates how it can work. Note that the first item (which is already known, see [13] and [20]) looks quite trivial. Anyhow, the statement was not so trivial and needed much efforts three years ago.

Example 3.9.   • *There exists a hyperbolic polytope with 4 horns.*
  • *We present a Laman-plus-one-graph (a rigidity circuit) embedded in S^2 as a pointed pseudotiling with 4 pseudo-di-gons.*
  Proof.
  The usual counts show that the spherical graph in Figure 3 is a rigidity circuit. It has a non-vanishing self-stress. The self-stress gives a virtual polytope, which is hyperbolic because the tiling is pointed. The number of horns equals 4 because it equals the number of di-gons (marked grey).   □

Obviously, the edge coloring of the fan of a hyperbolic virtual polytopes (= of an embedded self-stressed graph) reflects the sign pattern of the stress.

Question 3.10. *Given an embedded rigidity circuit, is it possible to detect the sign pattern of its (unique) self-stress by the combinatorics of the embedding (i.e. by corners information)?*

Question 3.11. *There seems to be no straightforward spherical analog for expansive motions [25]. Does there exist a parallel statement for spherically embedded graphs which exploits the same underlying reasons (duality together with mountain-valley arguments)?*
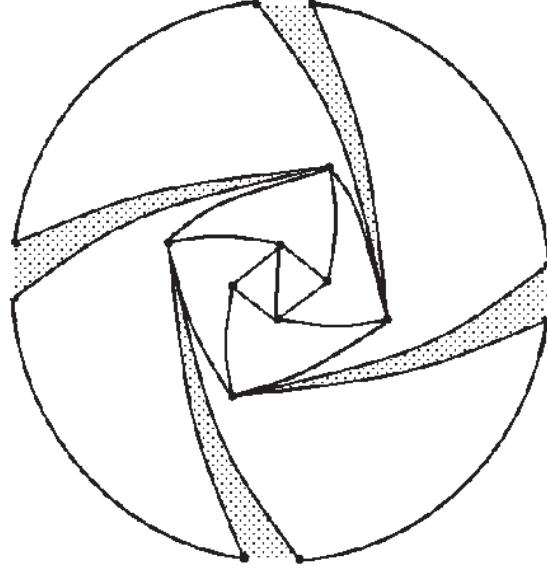
FIGURE 3.

A dictionary

Maxwell's reciprocals of a spherical self-stressed graph	virtual polytope
Maxwell's reciprocals of a **pointed** self-stressed graph	**hyperbolic** virtual polytope
pointed pseudo-tiling of the 2-dimensional sphere S^2	hyperbolic fan
a pseudo-tiling of S^2 with a non-zero self-stress	realizable hyperbolic fan
pseudo-di-gons of such a pseudo-tiling	horns of the hyperbolic polytope
3D lifting of a graph	graph of support function
negatively stressed edge of a spherical graph	blue edge of the hyperbolic fan
positively stressed edge of a spherical graph	red edge of the hyperbolic fan

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